Series 1

Exercise 1

The goal of this exercise is to derive the expression of the Green's function $g(\boldsymbol{r},\boldsymbol{r}')$ of the Helmholtz equation. As shown in class, this function satisfies the differential equation:

$$\nabla^2 g(\boldsymbol{r},\boldsymbol{r}') + k^2 g(\boldsymbol{r},\boldsymbol{r}') = -\delta(\boldsymbol{r}-\boldsymbol{r}')$$

1) In one sentence, give a good physical reason to justify that $g(\boldsymbol{r},\boldsymbol{r}')$ is actually a function of $\boldsymbol{r}-\boldsymbol{r}'$. Namely, $g(\boldsymbol{r},\boldsymbol{r}') = g(\boldsymbol{r}-\boldsymbol{r}')$.

As explained in class, $g(\boldsymbol{r},\boldsymbol{r}')$ is the potential vector field component generated at an observation point $\boldsymbol{r}$ by a source of unit strength located at $\boldsymbol{r}'$. $\boldsymbol{r}$ and $\boldsymbol{r}'$ are position vectors defined from the origin of the coordinate system. If g would depend on the precise value of these vectors, it would mean that the field would depend on our (arbitrary) choice of the origin of the coordinate system. This is not possible since the problem is translational invariant and does not depend on how far we are from an arbitrary origin. However, it may depend on the vector $\boldsymbol{r}-\boldsymbol{r}'$, **which is the same regardless of the choice of the origin of the coordinate system.**

2) In one sentence, give a good physical reason to justify that $g(\boldsymbol{r},\boldsymbol{r}')$ is actually a function of $|\boldsymbol{r}-\boldsymbol{r}'|$. Namely, we can go even further and write $g(\boldsymbol{r},\boldsymbol{r}') = g(|\boldsymbol{r}-\boldsymbol{r}'|)$.

Here, it is about rotational invariance of the problem. We are looking at a scalar field component generated by a scalar source in isotropic space. There is no way that this can depend on direction.

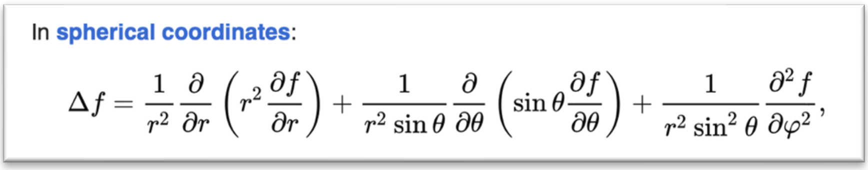
We will now note $r_1 = |\boldsymbol{r}-\boldsymbol{r}'|$.

3) By working in a spherical coordinate system, show that g satisfies

$$\frac{1}{r_1^2}\frac{d}{dr_1}\left[r_1^2\frac{dg}{dr_1}\right] + k^2 g = -\delta(r_1)$$

You may directly use the expression of ∇^2 found online at your favorite website. The sigularity of this equation at $r_1 = 0$ will be handled in a special way.

In the Wikipedia page on the "Laplace operator", under the section "Three dimensions", "Spherical coordinates", we find:



In **spherical coordinates**:

$$\Delta f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\varphi^2},$$

Note that $\nabla^2 = \Delta$. If we are in the spherical coordinate system of center $\boldsymbol{r}'$, then $r_1 = |\boldsymbol{r}-\boldsymbol{r}'|$ is the r coordinate of the above formula. And since $g(r_1)$ is not angle dependent, the angle derivatives are all zero. The equation is then obvious.

4) When $r_1 \neq 0$, show that the equation in step 3) reduce to:

$$\frac{d^2}{dr_1^2}[r_1 g] + k^2[r_1 g] = 0$$

Away from $r_1 = 0$, the delta distribution is zero by definition, and $\frac{1}{r_1^2}\frac{d}{dr_1}\left[r_1^2\frac{dg}{dr_1}\right] + k^2 g = 0$. We can then expand the derivative and find

$$\frac{1}{r_1^2}\left[r_1^2\frac{d^2 g}{dr_1^2} + 2r_1\frac{dg}{dr_1}\right] + k^2 g = \frac{d^2 g}{dr_1^2} + 2\frac{1}{r_1}\frac{dg}{dr_1} + k^2 g = 0$$

Multiplying on both sides by $r_1 \neq 0$, we get:

$$r_1\frac{d^2 g}{dr_1^2} + 2\frac{dg}{dr_1} + k^2 r_1 g = 0$$

Since $\frac{d^2}{dr_1^2}[r_1 g] = r_1\frac{d^2 g}{dr_1^2} + 2\frac{dg}{dr_1}$, we indeed find:

$$\frac{d^2}{dr_1^2}[r_1 g] + k^2[r_1 g] = 0$$

5) Show that the solution at $r_1 \neq 0$ is therefore:

$$g(r_1) = C\,\frac{e^{-jkr_1}}{r_1}$$

where C is a constant to be determined.

The equation in step 4) is an ordinary differential equation of second order, obeyed by $r_1 g$ and with constant coefficients. We solve it using standard methods. The roots of the characteristic polynomial are $\pm jk$. The general solution is therefore:

$$r_1 g(r_1) = Ce^{-jkr_1} + De^{jkr_1}$$

Only Ce^{-jkr_1} represents a wave flowing away from the source and is physically acceptable. We therefore must have $D = 0$, which proves the point.

6) The constant can be calculated by integrating the equation of step 3 over a spherical volume of radius ε, and taking the limit $\varepsilon \to 0$.
   a. Start with the right hand side. This is easy is you remember the definition of the Dirac distribution.
   b.

The definition of the Dirac distribution is that for any function $f(r_1)$, we have:

$$\int_{all\ space} f(r_1)\delta(r_1)dr_1 = f(r_1 = 0)$$

But integrating over a sphere S_ε or over all space is the same since the distribution $\delta(\boldsymbol{r})$ is zero everywhere outside of the sphere. In our case, $f(\boldsymbol{r}) = -1$ is constant, and we find:

$$\int_{S_\varepsilon} -\delta(r_1)dr_1 = -1$$

   c. Work on the left hand side and integrate over spherical shells of volume $4\pi r_1^2\,dr_1$, between 0 and ε. The singularity at $r_1 = 0$ disappears and the result after taking $\varepsilon \to 0$ should give:

$$C = 1/4\pi$$

There are two terms on the right hand side. Let's start with the first term:

$$\int_{S_\varepsilon} \frac{1}{r_1^2}\frac{d}{dr_1}\left[r_1^2\frac{dg}{dr_1}\right] d^3\boldsymbol{r_1} =$$

$$\int_{\mathbf{0}}^{\boldsymbol{\varepsilon}} 4\pi r_1^2 \frac{1}{r_1^2}\frac{d}{dr_1}\left[r_1^2\frac{dg}{dr_1}\right] dr_1 =$$

$$4\pi\int_{\mathbf{0}}^{\boldsymbol{\varepsilon}} d\left[r_1^2\frac{dg}{dr_1}\right] =$$

$$4\pi\varepsilon^2\frac{dg}{dr_1}(r_1 = \varepsilon) =$$

$$4\pi\varepsilon^2\frac{dg}{dr_1}(r_1 = \varepsilon) =$$

$$4\pi\varepsilon^2 Ce^{-jk\varepsilon}\left(\frac{-jk}{\varepsilon} - \frac{1}{\varepsilon^2}\right) =$$

$$-4\pi C \ [\varepsilon \to 0]$$

We have used the fact that $\frac{dg}{dr_1} = Ce^{-jkr_1}(\frac{-jk}{r_1} - \frac{1}{r_1^2})$.

Now, the second term.

$$\int_{S_\varepsilon} k^2 g(r_1)\, d^3\boldsymbol{r_1} =$$

$$\int_{\mathbf{0}}^{\boldsymbol{\varepsilon}} 4\pi r_1^2 k^2 g(r_1) dr_1 =$$

$$4\pi C\int_{\mathbf{0}}^{\boldsymbol{\varepsilon}} r_1^2 k^2 \frac{e^{-jkr_1}}{r_1} dr_1 =$$

$$4\pi C\int_{\mathbf{0}}^{\boldsymbol{\varepsilon}} k^2 r_1 e^{-jkr_1} dr_1 =$$

$$4\pi C\left[(1 + jkr_1)e^{-jkr_1}\right]_0^\varepsilon =$$

$$4\pi C\left[(1 + jk\varepsilon)e^{-jk\varepsilon} - 1\right] \ =$$

$$0 \ [\varepsilon \to 0]$$

Altogether, we have $-4\pi C + 0 = -1$, and therefore

$$C = 1/4\pi$$

For those interested, try to find the Green's function of the 2D Helmholtz equation. This is another story !

Exercise 2

We would like to calculate the exact electromagnetic field radiated by an infinitesimal dipole described by the current distribution:

$$\boldsymbol{J}(\boldsymbol{r}') = \boldsymbol{z}\, I\, L_0 \delta(\boldsymbol{r}')$$

where I is a current in Ampères and L_0 an effective length parameter.

1) Check that the unit of this current distribution is indeed A/m^2.

What's the unit of the Dirac delta $\delta(\boldsymbol{r}')$?

First of all, let us remind that $\delta(\boldsymbol{r}')$ is a short hand notation for $\delta(x')\delta(y')\delta(z')$. We also know that by definition,

$$\int \delta(u)du = 1$$

So, the sum of $\delta(u)du$ over all values of the real parameter u is a constant with no unit (namely 1). Thus $\delta(u)$, in general, has the inverse of the unit of du. When $du = dx'$ is a small length, we conclude that $\delta(x')$ has the unit of 1/m. Thus, $\delta(\boldsymbol{r}') = \delta(x')\delta(y')\delta(z')$ has the unit of 1/m^3, so $\boldsymbol{J}(\boldsymbol{r}')$ is in A.m/m^3=A/m^2.

2) Using the formulas on the last slide of the first course, compute the vector potential.

We start by noting that since we only have J_z, we only have A_z and $\boldsymbol{A} = A_z\boldsymbol{z}$. The formula for A_z given in the course is:

$$A_z = \mu_0 \int_V d^3\boldsymbol{r}'\; J_z(\boldsymbol{r}') \frac{e^{-jk|\boldsymbol{r}-\boldsymbol{r}\prime|}}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|}$$

and we must calculate it for $J_z(\boldsymbol{r}') = I\, L_0\delta(\boldsymbol{r}')$:

$$A_z = \frac{\mu_0}{4\pi} I\, L_0 \int_V d^3\boldsymbol{r}'\;\delta(\boldsymbol{r}') \frac{e^{-jk|\boldsymbol{r}-\boldsymbol{r}\prime|}}{|\boldsymbol{r}-\boldsymbol{r}'|}$$

By definition of the Dirac delta distribution:

$$\int_{all\ space} d^3\boldsymbol{r}'\;\delta(\boldsymbol{r}')f(\boldsymbol{r}') = f(\boldsymbol{0})$$

We get:

$$A_z = \mu_0 I\, L_0 \frac{e^{-jkr}}{4\pi r}$$

thus :

$$\boldsymbol{A} = \mu_0 I\, L_0 \frac{e^{-jkr}}{4\pi r}\boldsymbol{z}$$

3) Calculate the exact magnetic field $\boldsymbol{H}(\boldsymbol{r})$.
Hint: Formulas like $\nabla \times (f\boldsymbol{a}) = \nabla f \times \boldsymbol{a} + f\nabla \times \boldsymbol{a}$ can allow you to do the calculation fast.

$$\boldsymbol{H} = \frac{1}{\mu_0}\nabla \times \boldsymbol{A} = \frac{I\, L_0}{4\pi}\nabla \times \left(\frac{e^{-jkr}}{4\pi r}\boldsymbol{z}\right)$$

We use the proposed formula $\nabla \times (f\boldsymbol{a}) = \nabla f \times \boldsymbol{a} + f\nabla \times \boldsymbol{a}$ *with* $\boldsymbol{a} = \boldsymbol{z}$ *and* $f = \frac{e^{-jkr}}{4\pi r}$. $\boldsymbol{z}$ is a unit vector that does not depend on location, so $\nabla \times \boldsymbol{z} = \boldsymbol{0}$. We find:

$$\boldsymbol{H} = \frac{I\,L_0}{4\pi}\nabla\left(\frac{e^{-jkr}}{4\pi r}\right)\times\boldsymbol{z} = \frac{I\,L_0}{4\pi}\frac{\partial}{\partial r}\left(\frac{e^{-jkr}}{r}\right)\boldsymbol{r}\times\boldsymbol{z}$$

We can compute the cross product in spherical coordinates: $\boldsymbol{r}\times\boldsymbol{z} = \begin{pmatrix}1\\0\\0\end{pmatrix}\times\begin{pmatrix}\cos\theta\\-\sin\theta\\0\end{pmatrix} = -\sin\theta\,\boldsymbol{\varphi}$. We get:

$$\boldsymbol{H} = \frac{I\,L_0}{4\pi}\left(\frac{1}{r^2}+\frac{jk}{r}\right)e^{-jkr}\sin\theta\;\boldsymbol{\varphi}$$

4) Calculate the exact electric field $\boldsymbol{E}(\boldsymbol{r})$.
Hint: Two ways to go. Maxwell's equations or via ϕ. In case of emergency, stay seated, remain calm, and follow the flight attendants' instructions.

I chose here to go through Maxwell Faraday equation, written away from the point source:

$$\nabla\times\boldsymbol{H} = j\omega\varepsilon_0\boldsymbol{E}$$

or

$$\boldsymbol{E} = \frac{1}{j\omega\varepsilon_0}\nabla\times\boldsymbol{H} =$$

$$\frac{1}{j\omega\varepsilon_0}\frac{I\,L_0}{4\pi}\nabla\times\left(\left(\frac{1}{r^2}+\frac{jk}{r}\right)e^{-jkr}\sin\theta\;\boldsymbol{\varphi}\right) =$$

and we use again the formula[1] $\nabla\times(f\boldsymbol{a}) = \nabla f\times\boldsymbol{a} + f\nabla\times\boldsymbol{a}$

$$\frac{1}{j\omega\varepsilon_0}\frac{I\,L_0}{4\pi}\left\{\nabla\left(\left(\frac{1}{r^2}+\frac{jk}{r}\right)e^{-jkr}\sin\theta\right)\times\boldsymbol{\varphi} + \left(\left(\frac{1}{r^2}+\frac{jk}{r}\right)e^{-jkr}\sin\theta\right)\nabla\times\boldsymbol{\varphi}\right\}$$

Let's focus first on the gradient term inside the bracket (the Wikipedia page on spherical coordinates helps if, like us, you can't memorize all vector analysis formulas for gradient and curls):

$$\nabla\left(\left(\frac{1}{r^2}+\frac{jk}{r}\right)e^{-jkr}\sin\theta\right)\times\boldsymbol{\varphi} =$$

$$\frac{\partial}{\partial r}\left(\left(\frac{1}{r^2}+\frac{jk}{r}\right)e^{-jkr}\sin\theta\right)\boldsymbol{r}\times\boldsymbol{\varphi} + \frac{1}{r}\frac{\partial}{\partial\theta}\left(\left(\frac{1}{r^2}+\frac{jk}{r}\right)e^{-jkr}\sin\theta\right)\boldsymbol{\theta}\times\boldsymbol{\varphi} =$$

$$\left(\left(\frac{-2}{r^3}-\frac{jk}{r^2}\right)-jk\left(\frac{1}{r^2}+\frac{jk}{r}\right)\right)e^{-jkr}\sin\theta\,(-\boldsymbol{\theta}) + \left(\frac{1}{r^3}+\frac{jk}{r^2}\right)e^{-jkr}\cos\theta\,\boldsymbol{r} =$$

$$\left(\frac{1}{r^3}+\frac{jk}{r^2}\right)e^{-jkr}\cos\theta\,\boldsymbol{r} + \left(\frac{2}{r^3}+\frac{2jk}{r^2}-\frac{k^2}{r}\right)e^{-jkr}\sin\theta\,\boldsymbol{\theta}$$

---

[1] We did it this way here, but it's perhaps faster to use directly the expression of the curl in spherical coordinates. The path through the electric potential seems equally long since the divergence of A has to be calculated first, before taking a gradient, both in spherical coordinates.

Now the curl term:

$$\left(\left(\frac{1}{r^2}+\frac{jk}{r}\right)e^{-jkr}\sin\theta\right)\nabla\times\boldsymbol{\varphi}=$$
$$\left(\left(\frac{1}{r^2}+\frac{jk}{r}\right)e^{-jkr}\sin\theta\right)\left(\frac{1}{r\sin\theta}\cos\theta\ \boldsymbol{r}-\frac{1}{r}\boldsymbol{\theta}\right)=$$
$$\left(\frac{1}{r^3}+\frac{jk}{r^2}\right)e^{-jkr}\cos\theta\ \boldsymbol{r}-\left(\frac{1}{r^3}+\frac{jk}{r^2}\right)e^{-jkr}\sin\theta\ \boldsymbol{\theta}$$

Plugging back the two terms in the bracket to get $\boldsymbol{E}$:

$$\boldsymbol{E}=\frac{1}{j\omega\varepsilon_0}\frac{I\ L_0}{4\pi}\left\{\nabla\left(\left(\frac{1}{r^2}+\frac{jk}{r}\right)e^{-jkr}\sin\theta\right)\times\boldsymbol{\varphi}+\left(\left(\frac{1}{r^2}+\frac{jk}{r}\right)e^{-jkr}\sin\theta\right)\nabla\times\boldsymbol{\varphi}\right\}$$
$$=\frac{1}{j\omega\varepsilon_0}\frac{I\ L_0}{4\pi}\left\{\left(\frac{1}{r^3}+\frac{jk}{r^2}\right)e^{-jkr}\cos\theta\,\boldsymbol{r}+\left(\frac{2}{r^3}+\frac{2jk}{r^2}-\frac{k^2}{r}\right)e^{-jkr}\sin\theta\,\boldsymbol{\theta}\right.$$
$$\left.+\left(\frac{1}{r^3}+\frac{jk}{r^2}\right)e^{-jkr}\cos\theta\ \boldsymbol{r}-\left(\frac{1}{r^3}+\frac{jk}{r^2}\right)e^{-jkr}\sin\theta\ \boldsymbol{\theta}\right\}$$

We need to gather terms together, which simplifies things quite a bit:

$$\boldsymbol{E}=\frac{1}{j\omega\varepsilon_0}\frac{I\ L_0}{4\pi}\left\{\left(\frac{2}{r^3}+\frac{2jk}{r^2}\right)e^{-jkr}\cos\theta\,\boldsymbol{r}+\left(\frac{1}{r^3}+\frac{jk}{r^2}-\frac{k^2}{r}\right)e^{-jkr}\sin\theta\,\boldsymbol{\theta}\right\}$$

5) Can you identify in these fields which part is called the near field, which part is called the far field, and why ?

At short distances, terms in $\frac{1}{r^3}\gg\frac{1}{r^2}\gg\frac{1}{r}\gg 1$, so the near field are the $\frac{1}{r^3}$ terms only:

$$\boldsymbol{E_{near}}=\frac{1}{j\omega\varepsilon_0}\frac{I\ L_0}{4\pi r^3}e^{-jkr}\{2\cos\theta\,\boldsymbol{r}+\sin\theta\,\boldsymbol{\theta}\}$$
$$\boldsymbol{H_{near}}=\frac{I\ L_0}{4\pi}\frac{1}{r^2}e^{-jkr}\sin\theta\ \boldsymbol{\varphi}$$

At large distances, terms in $\frac{1}{r^3}\ll\frac{1}{r^2}\ll\frac{1}{r}$, so the far field are the $\frac{1}{r}$ terms only:

$$\boldsymbol{E_{far}}=j\frac{k^2}{\omega\varepsilon_0}\frac{I\ L_0e^{-jkr}}{4\pi r}\sin\theta\,\boldsymbol{\theta}$$
$$\boldsymbol{H_{far}}=jk\frac{I\ L_0e^{-jkr}}{4\pi r}\sin\theta\ \boldsymbol{\varphi}$$

Note that $\frac{k^2}{\omega\varepsilon_0}$ can also be written $\frac{k^2}{\omega\varepsilon_0}=\omega\mu_0=kZ_0$, so that the far fields satisfy the orthogonality condition typical of a plane wave:

$$\boldsymbol{H_{far}}=\frac{1}{Z_0}(\boldsymbol{r}\times\boldsymbol{E_{far}})$$

where $\boldsymbol{r}$ is the propagation direction of the spherical wave.